1972

A Mathematical Approach to the Forced Vibrations of the Suspended Compressor Block

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A MATHEMATICAL APPROACH TO

THE FORCED VIBRATIONS OF THE SUSPENDED COMPRESSOR BLOCK

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INTRODUCTION

When designing hermetic compressors for domestic applications, it is important to minimise the noise. The vibrations of the block caused by unbalance of the moving parts of a piston compressor may be an essential source of noise. It is the theoretical treatment of these vibrations we shall deal with below.

Consider the compressor block as a rigid body suspended by a number of linear springs. During operation the block executes undamped vibrations with six degrees of freedom. LAGRANGE's equations are used to set up the equations of motion. The eigenvalue problem of the system is solved, and it is shown, how the eigenvectors are used to decouple the equations, whereby these can be treated by linear operators.

THE EQUATIONS OF MOTION

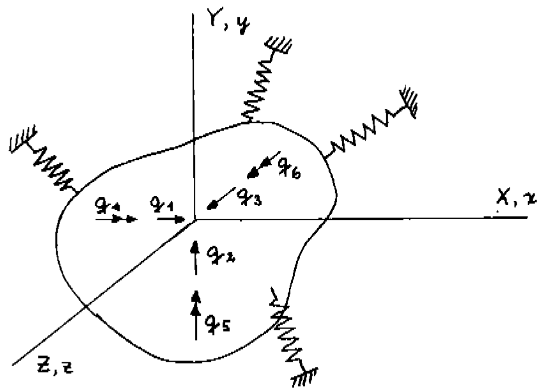

Figure 1

Consider a rigid body supported by a number of linear springs. With the body at rest two cartesian coordinate systems are imposed at the center of gravity of the body. The initial system is called X, Y, Z, the other system, which is fixed to the body, is called x, y, z.

The system has six degrees of freedom, v.i.z. three translations and three rotations, and in the initial system the position of the moving system are called (see figure 1)

Translations $q_1 \; q_2 \; q_3$ (⟶)

Rotations $q_4 \; q_5 \; q_6$ (⟶⟶)

The position (u_i, v_i, w_i) of a point (x_i, y_i, z_i) in the moving system can be written

$$(1) \quad (u_i, v_i, w_i) = (q_1, q_2, q_3) + (q_4, q_5, q_6) \times (x_i, y_i, z_i)$$
$$= (q_1 + q_5 z_i - q_6 y_i,\; q_2 + q_6 x_i - q_4 z_i,\; q_3 + q_4 y_i - q_5 x_i)$$

differentiating with respect to time gives

$$(2) \quad (\dot u_i, \dot v_i, \dot w_i) = (\dot q_1 + \dot q_5 z_i - \dot q_6 y_i,\; \dot q_2 + \dot q_6 x_i - \dot q_4 z_i,\; \dot q_3 + \dot q_4 y_i - \dot q_5 x_i)$$

Where $\dot q$ is the time derivative.

Kinetic energy

Let $V = V(x, y, z)$ be the volume of the body and $\varrho = \varrho(x, y, z)$ the density. The kinetic energy is

$$T = \frac{1}{2} \int_V \varrho (\dot u^2 + \dot v^2 + \dot w^2)\, dV$$

which gives

$$\begin{aligned} T = {} & \frac{1}{2} \int_V \varrho \{ \dot q_1^2 + \dot q_2^2 + \dot q_3^2 \}\, dV \\ & + \frac{1}{2} \int_V \varrho (y^2 + z^2) \dot q_4^2\, dV + \frac{1}{2} \int_V \varrho (x^2 + z^2) \dot q_5^2\, dV + \frac{1}{2} \int_V \varrho (x^2 + y^2) \dot q_6^2\, dV \\ & - \int_V \varrho x y \dot q_4 \dot q_5\, dV - \int_V \varrho x z \dot q_4 \dot q_6\, dV - \int_V \varrho y z \dot q_5 \dot q_6\, dV \\ & - \int_V \varrho (\dot q_6 y + \dot q_5 z) \dot q_1\, dV - \int_V \varrho (\dot q_6 x + \dot q_4 z) \dot q_2\, dV - \int_V \varrho (\dot q_5 x + \dot q_4 y) \dot q_5\, dV \end{aligned}$$

Because the moving system has origin at the center of gravity, the last three terms are equal to zero, and we have

$$(3) \quad \begin{aligned} T = \frac{1}{2} \{ & M (\dot q_1^2 + \dot q_2^2 + \dot q_3^2) \\ & + I_x \dot q_4^2 + I_y \dot q_5^2 + I_z \dot q_6^2 \\ & - 2 I_{xy} \dot q_4 \dot q_5 - 2 I_{xz} \dot q_4 \dot q_6 - 2 I_{yz} \dot q_5 \dot q_6 \} \end{aligned}$$

where M is the mass of the body and $I_x, I_y, I_z, I_{xy}, I_{xz}, I_{yz}$ are the moments of inertia.

Potential energy

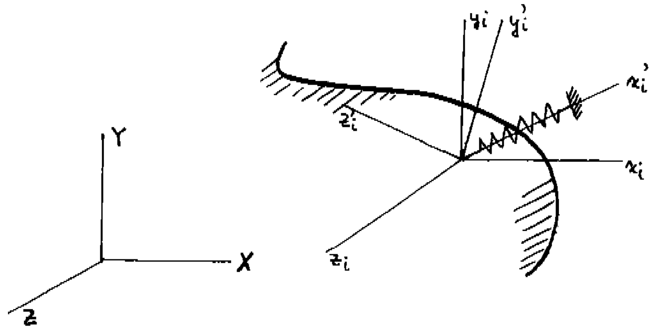

Figure 2

Assume that the stiffness matrix $[S]_i'$ of the individual spring is known in a element coordinate system x_i', y_i', z_i' . By a coordinate transformation the stiffness matrix is found in a coordinate system, which has the same directions as the initial system, but with origin at the suspension point of the spring

$$[S]_i = [R]_i\,[S]_i'\,[R]_i^*$$

where $[R]_i$ is the rotation matrix and $*$ denotes a transposed matrix.

The displacement of the suspension point of the spring is described by the column vector

$$\{D\}_i = \begin{Bmatrix} u_i \\ v_i \\ w_i \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

where u_i, v_i, w_i are obtained from (1). When displaced the potential energy ΔV_i of the spring is

$$\Delta V_i = \tfrac{1}{2}\,\{D\}_i^*\,[S]_i\,\{D\}_i$$

and if the body is supported by m springs, the total potential energy V can be written

$$(4) \qquad V = \tfrac{1}{2}\sum_{i=1}^{m}\{D\}_i^*\,[S]_i\,\{D\}_i$$

The equations of motion

Now we will apply LAGRANGE's equations for the energy expressions (3) and (4). Accordingly we have in the n-th direction

$$(5) \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_m}\right) - \frac{\partial T}{\partial q_m} + \frac{\partial V}{\partial q_m} = Q_m$$

where Q_m is the power function.

From the first term of (5), we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_1}\right) = M\ddot q_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_2}\right) = M\ddot q_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_3}\right) = M\ddot q_3$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_4}\right) = I_x\ddot q_4 - I_{xy}\ddot q_5 - I_{xz}\ddot q_6$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_5}\right) = -I_{xy}\ddot q_4 + I_y\ddot q_5 - I_{yz}\ddot q_6$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_6}\right) = -I_{xz}\ddot q_4 - I_{yz}\ddot q_5 + I_z\ddot q_6$$

The above equations can be written

$$(6) \qquad \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q}\right)\right\} = [M]\,\{\ddot q\}$$

where the inertia matrix is

$$[M] = \begin{bmatrix} M & & & & & \text{SYMMETRIC} \\ 0 & M & & & & \\ 0 & 0 & M & & & \\ 0 & 0 & 0 & I_x & & \\ 0 & 0 & 0 & -I_{xy} & I_y & \\ 0 & 0 & 0 & -I_{xz} & -I_{yz} & I_z \end{bmatrix}$$

and

$$\{\ddot q\} = \begin{Bmatrix} \ddot q_1 \\ \ddot q_2 \\ \ddot q_3 \\ \ddot q_4 \\ \ddot q_5 \\ \ddot q_6 \end{Bmatrix}$$

From the second term in (5), we obtain

$$\frac{\partial T}{\partial q_m} = 0 \quad \text{, because} \quad T = T(\dot q)$$

The last term on the left-side of equation (5) requires rather cumbersome calculations. The following example will show this. From (1) and (4) we obtain for the partial derivations in direction 1

$$\frac{\partial V}{\partial q_1} = \tfrac{1}{2}\sum_{i=1}^{m}\left[\{1\,0\,0\,0\,0\,0\}\,[S]_i \begin{Bmatrix} u \\ v \\ w \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_i + \{u\,v\,w\,q_4\,q_5\,q_6\}_i\,[S]_i \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}\right]$$

$$\frac{\partial V}{\partial q_1} = \frac{1}{2}\sum_{i=1}^{m}\left[S_{11}u + S_{12}v + S_{13}w + S_{14}q_4 + S_{15}q_5 + S_{16}q_6 + S_{11}u + S_{21}v + S_{31}w + S_{41}q_4 + S_{51}q_5 + S_{61}q_6\right]_i$$

and because $S_{jk} = S_{kj}$ in the stiffness matrix we have

$$\frac{\partial V}{\partial q_1} = \sum_{i=1}^{m}\left[S_{11}u + S_{12}v + S_{13}w + S_{14}q_4 + S_{15}q_5 + S_{16}q_6\right]_i$$

By substitution of (1), we obtain

$$\frac{\partial V}{\partial q_1} = \sum_{i=1}^{m}\left[S_{11}(q_1 + q_5 z - q_6 y) + S_{12}(q_2 + q_6 x - q_4 z) + S_{13}(q_3 + q_4 y - q_5 x) + S_{14}q_4 + S_{15}q_5 + S_{16}q_6\right]_i$$

$$= \sum_{i=1}^{m}\left[S_{11}q_1 + S_{12}q_2 + S_{13}q_3 + (S_{13}y - S_{12}z + S_{14})q_4 + (S_{11}z - S_{13}x + S_{15})q_5 + (S_{12}x - S_{11}y + S_{16})q_6\right]_i$$

which can be written

$$\frac{\partial V}{\partial q_1} = \{R_{11}\; R_{12}\; R_{13}\; R_{14}\; R_{15}\; R_{16}\} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

By similar calculations for the remaining coordinates, we obtain generally

$$\frac{\partial V}{\partial q_j} = \{R_{j1}\; R_{j2}\; R_{j3}\; R_{j4}\; R_{j5}\; R_{j6}\} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \qquad j = 2, \cdots, 6$$

and the last term on the left-hand-side of (5) can in matrix notation be written

(7) $$\left\{\frac{\partial V}{\partial q}\right\} = [R]\{q\}$$

where $[R]$ is the stiffness matrix of the system.

Collecting the power functions in a column vector and using (6) and (7), the equations of motion can be written

(8) $$[M]\{\ddot{q}\} + [R]\{q\} = \{Q\}$$

From the inertia matrix and the stiffness matrix it is seen, that we have a system of coupled differential equations. The equations are both statically and dynamically coupled because of the cross-terms of the stiffness- and inertia matrices. The equations can be solved numerically, but a decoupling of the equations will simplify the procedure considerably. Therefore it is necessary briefly to describe this decoupling.

DECOUPLING OF THE EQUATIONS

If we set $\{Q\} = \{0\}$ in (8), we obtain the equations of the free vibrations of the system

(9) $$[M]\{\ddot{q}\} + [R]\{q\} = \{0\}$$

We are interested in synchronous solutions, that is solutions separable in time and place

$$q_j(t) = u_j\, f(t) \qquad j = 1, \cdots, 6$$

Substitution in (9) gives

$$[M]\{u\}\ddot{f}(t) + [R]\{u\}f(t) = \{0\}$$

which can be written

$$\ddot{f}(t)\sum_{j=1}^{6} M_{jk}u_j + f(t)\sum_{j=1}^{6} R_{jk}u_j = 0 \qquad k = 1, \cdots, 6$$

or

$$-\frac{\ddot{f}(t)}{f(t)} = \frac{\sum_{j=1}^{6} R_{jk}u_j}{\sum_{j=1}^{6} M_{jk}u_j} \qquad k = 1, \cdots, 6$$

The right-hand-side is independent of time whereas the left-side is independent of the index k. Hereby both sides must be equal to a constant ω^2, and we have

(10) $$\ddot{f}(t) + \omega^2 f(t) = 0$$

(11) $$\sum_{j=1}^{6}(R_{jk} - \omega^2 M_{jk})u_k = 0 \qquad k = 1, \cdots, 6$$

The general solution of (10) is

$$f(t) = A\cos(\omega t + \varphi)$$

and we conclude that motions in which the coordinates execute harmonic vibrations are possible.

Equation (11) is a homogenous system of linear equations in u_i. A nontrivial solution is possible only if the determinant of the coefficient vanishes

$$\left| [R] - \omega^2 [H] \right| = 0$$

Expanding the determinant we obtain an equation of the six-th order in ω^2. It can be shown that the solutions $\omega_1^2, \omega_2^2 \cdots$ are positive and real. The solutions are called the eigenvalues of the system, and their positive square roots are the natural frequencies of the system.

Corresponding to each eigenvalue the equation (11) has a nontrivial vector solution, called the characteristic vector or eigenvector. In our case we have six eigenvectors and these are arranged in a square matrix

$$[u] = [\{u^{(1)}\}\{u^{(2)}\} \cdots \{u^{(6)}\}]$$

the so-called modal matrix.

It can be shown that the eigenvectors have a important property, they are orthogonal. This property can in matrix notation be expressed

$$(12) \qquad \{u^{(r)}\}[H]\{u^{(s)}\} = \begin{cases} m_{rs} & r = s \\ 0 & r \neq s \end{cases}$$

Since the set of eigenvectors are orthogonal and hence independent, they form a complete set of vectors in the sense, that they can be used as a basis for the decomposition of any vector $\{q(t)\}$, which represent a possible movement of the system. In view of this we can write

$$(13) \qquad \{q(t)\} = [u]\{\eta(t)\}$$

where $\{\eta(t)\}$ is a time-dependent column vector. It follows that

$$(14) \qquad \{\ddot{q}(t)\} = [u]\{\ddot{\eta}(t)\}$$

and substitution af (13) and (14) in the equations of motion (8) we have

$$[H][u]\{\ddot{\eta}\} + [R][u]\{\eta\} = \{Q\}$$

Multiplication by $[u]^*$, where $*$ denotes a transposition, gives

$$(15) \quad [u]^*[H][u]\{\ddot{\eta}\} + [u]^*[R][u]\{\eta\} = [u]^*\{Q\}$$

but because of the orthogonality conditions (12) we have

$$[u]^*[H][u] = \lceil m \rfloor$$

$$[u]^*[R][u] = \lceil r \rfloor$$

where $\lceil\ \rfloor$ denotes a diagonal matrix.

By substitution in (15) we get

$$(16) \qquad \lceil m \rfloor \{\ddot{\eta}\} + \lceil r \rfloor \{\eta\} = [u]^*\{Q\}$$

and it is seen, that the system of equations has been transformed to six decoupled differential equations.

Now the motion of the system can be found by applying the methods of linear operators to equation (16).

REFERENCES

1. CHEN, YU, Vibrations: Theoretical Methods, Addison-Wesley Publishing Comp., 1966.